Canonical Quantization and the Statistical Entropy of the Schwarzschild Black Hole

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Abstract

The canonical quantization of a Schwarzschild black hole yields a picture of the black hole that is shown to be equivalent to a collection of oscillators whose density of levels is commensurate with that of the statistical bootstrap model. Energy eigenstates of definite parity exhibit the Bekenstein mass spectrum, $M \sim \sqrt{\mathcal{N}} M_p$, where $\mathcal{N} \in \mathbf{N}$. From the microcanonical ensemble, we derive the statistical entropy of the black hole by explicitly counting the microstates corresponding to a macrostate of fixed total energy.

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1. Introduction

It has been recognized for some time that black holes behave as thermodynamic objects with a characteristic temperature and entropy and that these quantities are inherently quantum mechanical in nature.[1,2] This makes a clear understanding of the origins of black hole thermodynamics in terms of statistical principles one of the more interesting open problems, because a microscopic description of the black hole entropy requires a quantum theory of gravity so that detailed first-principle investigations concerning the black hole entropy should contribute toward a better understanding of how such a theory may be constructed and interpreted. The earliest attempt at a microscopic theory of black holes was due to Bekenstein[3], who concluded that the horizon area is the analog of an adiabatic invariant in mechanics. He then invoked the Christodoulou-Ruffini process[4] and the Bohr-Sommerfeld quantization rules to argue that the eigenvalues of the horizon area operator, $\hat{\mathcal{A}}$, of the black hole must be equally spaced, $\mathcal{A}_n \sim n l_p^2$. Dividing the horizon in cells of Planck area which get added one at a time, and assuming that each cell has the same (small) number of states, say k, he was able to derive the area law of black hole thermodynamics by estimating the number of microscopic states to be $\Omega \approx k^n$.

Bekenstein's hypothesis has led to many interesting attempts to derive the area quantization law and the black hole entropy from first principles. These attempts have mostly taken either the loop (or canonical) quantum gravity[5] approach or the string theory[6] approach. The ability to reproduce the Bekenstein-Hawking entropy from a microscopic counting of states must be considered a measure of the success of a candidate quantum theory of gravity. Thus, the successful derivation of the Bekenstein-Hawking entropy, for example, from a microcanonical ensemble of $D-$brane states[7] is generally considered a triumph for string theory and a necessary condition for string theory to be a convincing candidate for a theory of quantum gravity.

An approach to the entropy problem that makes explicit reference neither to

string theory nor to canonical quantum gravity is to count the number of states for a conformal field theory corresponding to the asymptotic symmetries of the black hole or to the symmetries of its horizon. This mechanism for calculating the number of microstates was recently proposed by Strominger[8] for the BTZ black hole.[9] Strominger used a result of Brown and Henneaux[10] which says that the asymptotic symmetry group of AdS_3 is generated by two copies of the Virasoro algebra with central charge $3l/2G$, and combined this result with Cardy's formula[11] to compute the asymptotic growth of states for this conformal field theory. A semi-classical analysis of a hot black hole suggests that the horizon is thermally oscillating while maintaining a fixed area. Recently, Carlip[12] has argued that Strominger's result is more generic than was originally believed, because the algebra of surface deformations of any black hole in any dimension contains a Virasoro algebra consisting of deformations that leave the horizon fixed. Cardy's formula once again yields the correct Bekenstein-Hawking entropy. In this approach, the horizon is treated as a boundary and all of the relevant degrees of freedom of the black hole are assumed to lie on it. The states themselves are not explicitly displayed.

The results from loop quantum gravity have been no less dramatic. Here spin network states are used as a complete, orthonormal basis in the Hilbert space and loop microstates are explicitly counted using techniques developed in ref.[13]. In one approach[14], loop quantum gravity is used to compute the microstates of the horizon, holding the area fixed. In another approach, one analyzes the classical theory outside, treating the horizon as a boundary. Quantization yields surface states which are counted by an effective Chern-Simons theory on the boundary.[15] Both approaches yield an entropy proportional to the area of the horizon, but the proportionality constant is a free, finite and dimensionless parameter, not determined by the theory and the horizon area eigenvalues are not equally spaced.

In this article we take a midi-superspace approach to the canonical quantization of the Schwarzschild black hole and show that it leads naturally to Bekenstein's mass quantization law (equally spaced area eigenvalues) as well as to the area law

of black hole entropy as computed from a genuine microcanonical ensemble. The quantization leads to an amusing picture of a black hole which is akin to statistical bootstrap models for hadrons, whose statistical mechanics was studied many years ago by Frautschi[16] and Carlitz[17] . We will use similar techniques to study the statistical properties of the Schwarzschild black hole, but our starting point will be a recent solution[18] of the Wheeler-DeWitt equation in terms of variables first introduced by Kuchař and Brown.[19,20]

2. Quantization

As we have previously shown[18], combining the Hamiltonian reduction of spherical geometries due to Kuchař[19] and the coupling to dust as proposed by Brown and Kuchař[20] allows for the derivation of a simple, decoupled (Wheeler-DeWitt) equation describing the Schwarzschild black hole. The non-rotating dust is introduced in such a way that its role is only as a time keeper. Consider the gravity-dust system,

$$S \; = \; -\frac{1}{16\pi}\int d^4x\sqrt{-g}\mathcal{R} \; - \; \frac{1}{2}\int d^4x\sqrt{-g}\epsilon(x)\left[g_{\alpha\beta}U^\alpha U^\beta + 1\right], \qquad (2.1)$$

in the general spherically symmetric spacetime,

$$ds^2 \; = \; N^2dt^2 \; - \; L^2(dr - N^r dt)^2 \; - \; R^2 d\Omega^2, \qquad (2.2)$$

where $N(t,r)$ and $N^r(t,r)$ are respectively the lapse and shift functions, $R(t,r)$ is the physical radius or curvature coordinate, $\epsilon(t,r)$ is the density of the collapsing dust in its proper frame, $\mathcal{R}$ is the scalar curvature and U^α are the components of the dust velocity. The action, S, in (2.1), may be be recast into the form

$$S \; = \; \int dt dr \left[P_L\dot{L} \; + \; P_R\dot{R} \; + \; P_\tau\dot{\tau} \; - \; NH \; - \; N^r H_r\right]$$
$$+ \;\; surface \; terms, \qquad (2.3)$$

where we have introduced, following Brown and Kuchař,[20] the dust proper time variable, τ, which in general will serve as extrinsic time. P_L and P_R are the

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momenta conjugate to L and R respectively, and the super Hamiltonian, H, and super momentum, H_r, are respectively given by

$$H = -\left[\frac{P_L P_R}{R} - \frac{L P_L^2}{2R^2}\right] + \left[-\frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{RR'}{L}\right)'\right]$$
$$+ P_\tau \sqrt{1 + \tau'^2/L^2} \qquad (2.4)$$

and

$$H_r = R'P_R - LP_L' + \tau'P_\tau, \qquad (2.5)$$

where the prime denotes a derivative with respect to the ADM label coordinate r. The constraints in the above form do not "decouple" and are very difficult to resolve as they stand. However, from the general system in (2.2), Kuchař[19] showed how one can pass by a canonical transformation to a new canonical chart with coordinates M and R together with their conjugate momenta, P_M and $\overline{P}_R$, where M is the Schwarzschild "mass" and R is the curvature coordinate. In this system the constraints are greatly simplified and the phase space variables have immediate physical significance. The canonical transformation is well-defined as long as the metric obeys standard fall-off conditions[19] and, as long as these fall-off conditions are obeyed, the surface action can be recast in the form

$$surface\ terms = \int dt\,[\pi_+\dot{\tau}_+ + \pi_-\dot{\tau}_- - N_+C_+ - N_-C_-], \qquad (2.6)$$

where $\tau_\pm$ are the proper times measured on the parametrization clocks at right (left) infinity. The constraints $C_\pm = \pm\pi_\pm + M_\pm$ identify their conjugate momenta as the mass at right (left) infinity. In terms of the new variables the entire action, along with the surface term is

$$S = \int dt \int dr \left[\overline{P}_M \dot{M} + \overline{P}_R \dot{R} + \overline{P}_\tau \dot{\tau} - NH - N^r H_r\right], \qquad (2.7)$$

where $\overline{P}_M = P_M - \tau'$, $\overline{P}_\tau = P_\tau + M'$ and P_M and P_τ are the original Brown-Kuchař variables. The transformations leading up to these variables may be found

in ref.[19]. In passing to the transformed momenta, $\overline{P}_\tau$ and $\overline{P}_M$, we have made a canonical transformation[18] generated by $M\tau'$, which effectively absorbs the surface terms. (Thus, we have implicitly fixed the dust proper time to coincide at infinity with the parametrization clocks.)

The super Hamiltonian and super momentum constraints become

$$
\begin{aligned}
H &= -\left[\frac{F^{-1}M'R' + F\overline{P}_R(\overline{P}_M + \tau')}{L}\right] \\
&\quad + (\overline{P}_\tau - M')\sqrt{1 + \tau'^2/L^2} = 0
\end{aligned}
\tag{2.8}
$$

and

$$
H_r = M'\overline{P}_M + R'\overline{P}_R + \tau'\overline{P}_\tau = 0, \tag{2.9}
$$

where we have used

$$
L^2 = F^{-1}R'^2 - F(\overline{P}_M + \tau')^2 \tag{2.10}
$$

and $F = 1 - 2M/R$. L^2, being the component g_{rr} of the spherically symmetric metric in (2.2), must be positive definite everywhere. F is positive in the exterior (Schwarzschild) region and negative in the interior and this will play an important role in the consistency conditions that follow. By direct computation of Poisson brackets, it is easy to determine the "velocities" in terms of the conjugate momenta from the above expressions and they are

$$
\begin{aligned}
\dot{\tau} &= N\sqrt{1 + \tau'^2/L^2} + N^r\tau', \\
\dot{R} &= -\frac{NF(\overline{P}_M + \tau')}{L} + N^r R', \\
\dot{M} &= \frac{NR'\tau'(\overline{P}_\tau - M')}{L^3} + N^r M'.
\end{aligned}
\tag{2.11}
$$

The constraints in (2.8) and (2.9) generate the local symmetries of the theory. In the Schroedinger representation, they correspond to functional differential equations that the state functional must satisfy, i.e., the canonical variables must be

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raised to operator status and the constraints considered as operator constraints on the (Wheeler-DeWitt) wave functional $\psi_{WD}[\tau, R, M]$,

$$\hat{H}\psi_{WD}[\tau, R, M] \;=\; 0 \;=\; \hat{H}_r\psi_{WD}[\tau, R, M]. \tag{2.12}$$

These equations are, in fact, an infinite set of equations, one for each spatial point on the spatial hypersurface.

If the super momentum constraint in (2.9) is used to eliminate $\overline{P}_M$ in the expression for the super Hamiltonian in (2.8), the latter constraint turns into[18]

$$(\overline{P}_\tau - M')^2 \;+\; F\overline{P}_R^2 \;-\; \frac{M'^2}{F} \;=\; 0. \tag{2.13}$$

We will now specialize to the black hole by requiring $M' = 0$ so that only the homogeneous mode of $M(t, r)$ survives. In this way the dust is made tenuous and the proper time variable must be thought of as the proper time of a *test* particle (an ideal clock) in free fall. To understand its role in the canonical reduction, recall that the Schwarzschild metric in comoving coordinates can be viewed as a special case of a marginally bound Tolman-Bondi metric for the collapse of inhomogeneous dust, which has the general form

$$ds^2 \;=\; d\tau^2 \;-\; R'^2(\tau, \rho)d\rho^2 \;-\; R^2(\tau, \rho)d\Omega^2, \tag{2.14}$$

where τ is the dust proper time and the prime denotes a derivative with respect to ρ. The curvature coordinate, $R(\tau, \rho)$, is obtained from Einstein's equations, after an appropriate scaling[21], as

$$R^{3/2} \;=\; \rho^{3/2} \;-\; \frac{3}{2}\sqrt{F(\rho)}\tau, \tag{2.15}$$

in terms of an arbitrary function (the "mass" function), $F(\rho)$. It represents the mass contained within a shell of radius ρ. The energy density of the dust is given

by

$$\epsilon(\tau, \rho) \quad = \quad \frac{F'}{R^2 R'}.$$ (2.16)

In particular, if $F(\rho) = a^2$ (const.), the energy density vanishes everywhere except at the singularity, $R = 0$, and the metric in (2.14) describes a (Schwarzschild) black hole of mass $a^2/2$. This follows from the coordinate transformation[22]

$$\begin{aligned}
\tau &= T + a \int \frac{\sqrt{R} dR}{R - a^2} \\
\rho^{3/2} &= R^{3/2} + \frac{3}{2} a \left(T + a \int \frac{\sqrt{R} dR}{R - a^2} \right),
\end{aligned}$$ (2.17)

which takes (2.14) into the standard Schwarzschild form, with $2M = a^2$ and where T is the Killing time. This spacetime transformation may be re-expressed as a point transformation on the phase space.

The resulting Wheeler-DeWitt equation (with $M' = 0$) is decoupled, and the quantum state may be expressed formally as a direct product of states defined at each spatial point,

$$|\psi_{WD}\rangle \quad = \quad \prod_r |\Psi_r\rangle,$$ (2.18)

where we have used r as a label. Each of the $|\Psi_r\rangle$ is normalized w.r.t. a suitable measure in the space of functions $\tau(r), R(r), M$ at fixed r and the system reduces to a set of *independent* Schroedinger equations, one for each spatial point. The wave functional is given by

$$\begin{aligned}
\psi_{WD}[\tau, R, M] &= \prod_r \Psi_r[\tau(r), R(r), M] \\
&= \Psi_1 \otimes \Psi_2 \otimes ... \otimes ... \otimes \Psi_N.
\end{aligned}$$ (2.19)

The second equation is written by imagining that a lattice is placed on each spatial hypersurface so that the classically continuous label coordinate, r, is discretized. In this form, the Wheeler-DeWitt wave functional represents a collection of, say, N

decoupled systems, each determined by the same Schroedinger equation and obeying the same boundary conditions. The precise value of N cannot be determined at this level. We will ascertain its value by requiring it to maximize the density of states.

The second constraint enforces spatial diffeomorphism invariance of the wave functional on hypersurfaces orthogonal to the dust proper time. By taking functional derivatives of ψ_{WD} in (2.19), it is easy to see that this constraint enforces $\Psi'_r[\tau(r), R(r), M] = 0$, where the prime denotes a derivative with respect to the label coordinate r.

We also see, from eq. (2.13), that, for every label r, the (Schroedinger) equation reads

$$\nabla^2 \Psi \;\; = \;\; \gamma^{ab} \nabla_a \nabla_b \Psi \;\; = \;\; \tilde{H} \Psi \;\; = \;\; 0, \tag{2.20}$$

where γ_{ab} is the field space metric, $\gamma_{ab} = \mathrm{diag}(1, 1/F)$, and ∇_a is the covariant derivative with respect to this metric. The operator ∇^2 is the Laplace-Beltrami operator having γ_{ab} as its covariant metric, and eq. (2.20) is a massless "Klein-Gordon" equation. It is hyperbolic in the region $R < 2M$ (the interior of the Kruskal manifold) but elliptic in the region $R > 2M$ (the exterior). This is because the quantity F is negative in the interior, but positive in the exterior. As shown in ref. [18], this means that the unique positive energy solution of the Wheeler-DeWitt equation in the exterior, that is compatible with spatial diffeomorphism invariance, is identically zero. The dynamics is therefore confined to the *interior* of the hole. This is consistent with the assumed geometry, as the asymptotic observer sees the exterior region of the spacetime as static.

In the interior, the Schroedinger equation is hyperbolic and it is convenient to transform to the coordinate $\overline{R}_*$ defined by

$$\overline{R}_* \;\; = \;\; -\sqrt{R(2M - R)} \; + \; M \tan^{-1}\left[\frac{R - M}{\sqrt{R(2M - R)}}\right]. \tag{2.21}$$

The new coordinate lies in the range $(-\frac{\pi M}{2}, +\frac{\pi M}{2})$ and the wave equation,

$$\partial_\tau^2 \Psi \;-\; \partial_*^2 \Psi \;\;=\;\; 0, \tag{2.22}$$

now defines the quantum theory whose Hilbert space is $\mathcal{H} := \mathcal{L}^2(\mathbf{R}, d\overline{R}_*)$ with inner product

$$\langle \Psi_1, \Psi_2 \rangle \;\;=\;\; \int\limits_{-\frac{\pi M}{2}}^{+\frac{\pi M}{2}} d\overline{R}_* \Psi_1^\dagger \Psi_2 \;. \tag{2.23}$$

The general (positive energy) solution is

$$\Psi_{in} \;\;=\;\; c_+(M)e^{-iE(\tau+\overline{R}_*)} \;\;+\;\; c_-(M)e^{-iE(\tau-\overline{R}_*)} \tag{2.24}$$

where $c_\pm$ are functions only of M. We must impose the super momentum constraint, which reads

$$(\tau' + R')c_+(M)e^{-iE(\tau+\overline{R}_*)} \;\;+\;\; (\tau' - R')c_-(M)e^{-iE(\tau-\overline{R}_*)} \;\;=\;\; 0, \tag{2.25}$$

assuming $E > 0$. A consistent and physically meaningful solution to this equation is $\tau' = \overline{R}'_* = 0$. Returning to (2.11), we see that the choice implies that $\dot{\tau} = N$ and $\dot{M} = 0$. Setting $N = 1$, the dust proper time turns into the asymptotic Minkowski time and the energy, E, should be associated with the ADM mass of the black hole.

Imposing continuity across the horizon, this solution will match the solution in the exterior ($\Psi = 0$) at $R = 2M$, if

$$c_-(M) \;\;=\;\; -\, c_+(M)e^{-iEM\pi}, \tag{2.26}$$

so that the solution in the interior is now of the form[18]

$$\Psi_{in} \;\;=\;\; c_+(M) \left[e^{-iE(\tau+\overline{R}_*)} \;-\; e^{-iEM\pi} e^{-iE(\tau-\overline{R}_*)} \right] \;. \tag{2.27}$$

There does not seem to be a natural way to impose further boundary conditions. Certainly, boundary conditions cannot be imposed at the classical singularity where

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the canonical reduction will break down anyway. Nevertheless, one notes that the parity operator $\overline{R}_* \to -\overline{R}_*$ commutes with the Hamiltonian. States of definite parity will vanish at the classical singularity and exhibit a discrete spectrum given by

$$
\begin{aligned}
\Psi_{in}^{(+)} &= \frac{1}{\sqrt{\pi M}} e^{-iE\tau} \cos E\overline{R}_* & EM &= (2n+1) \;, \\
\Psi_{in}^{(-)} &= \frac{1}{\sqrt{\pi M}} e^{-iE\tau} \sin E\overline{R}_* & EM &= 2n.
\end{aligned}
\tag{2.29}
$$

Furthermore, as the dust proper time is identified with the asymptotic Minkowski time and the total energy with the ADM mass of the black hole, we are led to the Bekenstein mass quantization rule

$$
M_n = \sqrt{n} M_p
\tag{2.30}
$$

for the definite parity states.

States of indefinite parity do not admit a quantized mass spectrum, but there are three good reasons to confine attention to states of definite parity. Firstly, because the parity operator commutes with the hamiltonian, states of definite parity are guaranteed to remain so for all time, which is in harmony with our intuitive notion of an "eternal" black hole. Secondly, definite parity eigenstates do not support the singularity at the origin, which, given that the entire canonical quantization program breaks down there, is an attractive feature. Finally, and perhaps most importantly, we shall count only the definite parity states in what follows and show that they fully account for the entropy of the black hole.

3. The Entropy

To compute the entropy, we must enumerate the states of the system. To this effect, it is convenient to reformulate the problem by recognizing that the wave

equation at each label, r, in (2.20) is derivable from the action,

$$S = -\frac{1}{2}\sum_{r=1}^{N}\int_{\mathcal{R}} d^2X \sqrt{|\gamma|}\gamma^{ab}\partial_a\Psi_r^\dagger\partial_b\Psi_r, \tag{3.1}$$

where $X \in (\tau, \overline{R}_*)$, the integral is over the interior of the Kruskal manifold and such that $\Psi_r(\tau, \overline{R}_*) = 0$ at $\overline{R}_* = -\pi GM/2, +\pi GM/2, \ \forall \ r$. Imposition of these boundary conditions automatically confines attention to states of definite parity. Recall that the Wheeler-DeWitt wave functional is a direct product state, as given in (2.19). As we will see, each component of the direct product can be thought of as describing a tower of oscillators living in the internal space parametrized by the phase space coordinates $(\tau, \overline{R}_*)$.

Using (3.1), performing a mode expansion of Ψ_r and combining both even and odd parities, we can express the contribution of any one of the lattice sites to the total energy of the system in terms of pairs α_n, β_n of creation and annihilation operators as follows

$$\hat{\mathcal{H}}_r = \frac{M_p^2}{M}\sum_{n_r}(\alpha_{n_r}^\dagger\alpha_{n_r} + \beta_{n_r}^\dagger\beta_{n_r}), \tag{3.2}$$

where

$$\begin{aligned}{}[\alpha_{n_r}, \alpha_{n_r}^\dagger] &= n_r \\ [\beta_{n_r}, \beta_{n_r}^\dagger] &= n_r \end{aligned} \tag{3.3}$$

are the only non-vanishing commutators. At each site, r, one therefore has a hierarchy of two dimensional oscillators. The total energy is the sum over contributions from each of the N sites, i.e.,

$$\hat{H}_{tot} = \frac{M_p^2}{M}\sum_{r=1}^{N}\sum_{n_r}(\alpha_{n_r}^\dagger\alpha_{n_r} + \beta_{n_r}^\dagger\beta_{n_r}), \tag{3.4}$$

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which gives (the total energy is the mass of the black hole)

$$M = \frac{1}{M} \sum_r m_r^2$$

$$= \frac{M_p^2}{M} \sum_{r=1}^{N} \sum_{n_r, l_r} (n_r N_{n_r} + l_r K_{l_r}) \qquad (3.5)$$

$$\rightarrow \quad M = \sqrt{\mathcal{N}} M_p, \quad \mathcal{N} \in \mathbf{N} \cup \{0\},$$

where n_r, l_r and N_{n_r}, K_{l_r} are respectively the level number and the occupation number at level number $n_r(l_r)$, corresponding to the oscillators at site r.

Let $\rho_D(m_r)$ be the density of levels describing each site, r. This is just the number of states with mass given by $m_r^2 = \nu_r M_p^2$ ($\nu_r \in \mathbf{N}$), and is known to have the asymptotic ($m_r >> M_p$) form[23]

$$\rho_D(m_r) = c \times m_r^{-(D+1)/2} \times \exp\left[2\pi\sqrt{\frac{D}{6}}\frac{m_r}{M_p}\right], \qquad (3.6)$$

where c is a constant and D is the dimension of the oscillator. For a generic level density, $\rho_D(m_r)$, the density of states may be written as

$$\Omega(N, M) = \prod_{r=1}^{N} \int_{M_0}^{\infty} dm_r \rho_D(m_r) \delta\left(\frac{1}{M}\sum_{s=1}^{N} m_r^2 - M\right), \qquad (3.7)$$

where M_0 is the lowest value of m_r for which the density of levels is valid. As each site is fixed, there are no further phase space integrals. The delta function in (3.7) imposes energy conservation as required by (3.5). Let us assume that $\rho_D(m_r)$ is such that the dominant contribution to the mass integrals comes from states with large mass (unless these states are forbidden by energy conservation). This is certainly true for the level density in (3.6). Then define the quantity

$$\sigma_N(M) = \int_{NM_0}^{M} dx \prod_{r=1}^{N} \int_{M_0}^{\infty} dm_r \rho_D(m_r) \delta(\frac{1}{M}\sum_r m_r^2 - x), \qquad (3.8)$$

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in terms of which we may write

$$\Omega(N, M) \ = \ \frac{d}{dM}\sigma_N(M). \tag{3.9}$$

The $\delta-$ function restricts the limits of the mass integrals in the definition of $\sigma_N(M)$, they no longer run to infinity. Following Carlitz[17], we estimate the rth integral by

$$\int\limits_{M_0}^{\sqrt{M\Lambda_r(x)}} dm_r\rho_D(m_r), \tag{3.10}$$

provided that the $\Lambda_r(x)$ are subject to the constraint

$$\sum_{r=1}^{N}\Lambda_r(x) \ = \ x \ . \tag{3.11}$$

The maximum contribution to $\sigma_N(M)$ is obtained when the $\Lambda_r(x)$ are all of the order of x/N. This provides an estimate for the integrals in (3.8), and one finds, quite generally,

$$\Omega(N, M) \ = \ a^N f(\xi)^N, \tag{3.12}$$

where $\xi = bM^2/NM_p^2$, and a and b are dimensionless constants . The most probable number of sites is obtained by maximizing $\Omega(N, M)$ with respect to N. Therefore, consider

$$\frac{\partial}{\partial N}\ln\Omega(N, M) \ = \ \ln a \ + \ \ln f(\xi) \ - \ \xi\frac{\partial}{\partial\xi}\ln f(\xi) \ = \ 0, \tag{3.13}$$

which depends only on ξ. Assume that Ω is maximized for some value, say α^{-1}, of ξ. Then this value gives the corresponding number of sites as

$$N_{max} \ = \ \alpha b\frac{M^2}{M_p^2}. \tag{3.14}$$

N_{max} is therefore proportional to the area of the horizon. A remarkable consequence is that all the degrees of freedom can be *treated as though* they resided just there, that is on the horizon itself.

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Equation (3.13) can be integrated and the solution written in terms of a single dimensionless constant, γ. One finds that

$$e^{-\gamma\xi}f(\xi) = \frac{1}{a}. \tag{3.15}$$

This constant, γ then determines the maximum number of states according to

$$
\begin{aligned}
\ln\Omega_{max}(M) &= N\ln(a) + N\ln f(\xi)|_{N_{\max}} \\
&= N\ln e^{-\gamma\xi}f(\xi) + N\ln(a) + N\gamma\xi|_{N_{max}} \\
&= N\gamma\xi|_{N_{max}} = b\gamma\frac{M^2}{M_p^2}
\end{aligned} \tag{3.16}
$$

and is, itself, to be determined from (3.15) at $\xi = \alpha^{-1}$, $i.e.$,

$$\gamma = \alpha\ln[af(\alpha^{-1})]. \tag{3.17}$$

One thus recovers the entropy

$$S = \ln\Omega_{max}(M) = \frac{b\gamma}{4\pi l_p^2}\left(\frac{\mathcal{A}}{4}\right), \tag{3.18}$$

where $\mathcal{A}$ is the horizon area. This is the area law, provided that $\gamma > 0$, and it is independent of the precise form of the density of levels (except for proportionality constants), only requiring that the latter is such that the dominant contributions come from the most massive states permitted by energy conservation. S inherits its dependence on M only from the dispersion relation in (3.5).

As an example, let us use the density of levels as given in (3.6) although we will see that it is strictly not correct to do so. In the present context $D = 2$, giving

$$
\begin{aligned}
\rho_2(m_r) &= c \times m_r^{-3/2} \times \exp\left[\frac{2\pi}{\sqrt{3}}\frac{m_r}{M_p}\right] \\
\Omega(N, M) &= c^N \prod_{r=1}^{N}\int_{M_0}^{\infty} dm_r m_r^{-3/2} e^{2\pi m_r/\sqrt{3}M_p}\delta(\frac{1}{M}\sum_r m_r^2 - M).
\end{aligned} \tag{3.19}
$$

Performing all the necessary steps, one recovers (3.12) with

$$f(\xi) = \left[\sqrt{\pi} \mathrm{Erfi}(\xi^{1/4}) - \sqrt{\pi} \mathrm{Erfi}(\xi_0^{1/4}) - \frac{e^{\xi^{1/2}}}{\xi^{1/4}} + \frac{e^{\xi_0^{1/2}}}{\xi_0^{1/4}} \right], \qquad (3.20)$$

where $\mathrm{Erfi}(z) = \mathrm{Erf}(iz)/i$ is the imaginary error function,

$$\begin{aligned} \xi &= \left(\frac{4\pi^2 M^2}{3 N M_p^2} \right) \qquad \to \qquad b = \frac{4\pi^2}{3} \\ \xi_0 &= \left(\frac{4\pi^2 M_0^2}{3 M_p^2} \right) \end{aligned} \qquad (3.21)$$

and

$$a = \sqrt{\frac{8\pi c^2}{\sqrt{3} M_p}}. \qquad (3.22)$$

Therefore one has,

$$S = \frac{\pi \gamma}{3 l_p^2} \left(\frac{\mathcal{A}}{4} \right). \qquad (3.23)$$

The constant γ will depend on M_0, the lowest value of the mass for which the density of levels, $\rho_2(m_r)$, in (3.6) is valid, and on the constant, c. We have taken $c = \sqrt{M_p}$, $M_0 = M_p$ and found $\gamma \approx 0.089$. γ is found to decrease sharply with increasing M_0/M_p. This behavior contrasts with the expected value of $\pi\gamma/3 \approx 1$. The reason for this discrepancy may be traced to our use of the *asymptotic* level density. Recall that $m_r^2 \le M^2/N$ and we found, quite generally, that the density of states was maximized when $N \sim M^2/M_p^2$. This implies that $m_r \sim M_p$, which contradicts our use of (3.6).

When only the lowest levels are occupied, we can approximate $\rho_D(m_r)$ by a constant, say k/M_p. This gives $f(\xi) = \xi^{1/2}$, $b = k^2$ and $a = 1$, and inserting these values into (3.17) and (3.18) yields

$$S = \frac{k^2}{8\pi e l_p^2} \left(\frac{\mathcal{A}}{4} \right). \qquad (3.24)$$

To recover Hawking's temperature we must take $k \sim \sqrt{8\pi e}$.

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4. Discussion

We have shown that the canonical reduction of spherical geometries due to Kuchař[19], combined with the introduction of an extrinsic time variable via the coupling to non-rotating dust as proposed by Kuchař and Brown[20], leads to a remarkably simple description of the eternal Schwarzschild black hole. The Wheeler-DeWitt wave functional was seen to be expressible as a direct product state, which was interpreted as a collection of N oscillator hierarchies, each of which could be described by a free, massless, complex scalar propagating in a "flat" two dimensional background (the internal, "metric" space) and confined to the interior of the Kruskal manifold. The canonical quantization program does not allow for an estimate of N, but, using techniques from the study of the statistical behavior of dual models, the Veneziano model and the statistical bootstrap model by Frautschi[16] and Carlitz[17], we discovered that the value of N that maximizes the density of states is given by $N \sim M^2/M_p^2$, i.e., the number of (lattice) sites is proportional to the area of the horizon, implying that all the degrees of freedom may be thought of as residing on the horizon itself. It is a remarkable result that probably has a deeper meaning. One is tempted to speculate, for example, that this, or some similar mechanism, may be a consequence of, or indeed a justification for, some form of a "holographic" principle. Moreover, because the total energy, M, of the black hole, is divided between these N lattice sites according to the dispersion relation $\sum_r m_r^2 = M^2$, each oscillator is virtually in its ground state, with a small associated degeneracy. The picture that emerges thus coincides also with Bekenstein's original way[3] of estimating the entropy by dividing the horizon into cells of Planck area, each of which has a small number of associated states.

We were able to calculate the statistical entropy of the black hole by evaluating the microcanonical density of states of the system. Thus we recovered the area law of black hole thermodynamics, which is seen to be the consequence of the dispersion relation in (3.5) and the area quantization rule. It is noteworthy that all the states are explicitly displayed, both the mass quantization rules and the statistical entropy are recovered, and no explicit appeal to boundary states has had to be made in

this approach.

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